Existence or non existence of ground states for NLS on doubly periodic metric graphs: a dimensional crossover

I. Variational problems for Non-linear Schrödinger equations (NLS) on metric graphs

Ref: R. Adami, E. Serra, P. Tilli, Nonlinear dynamics on branched structures and networks

(tNLS) $\quad -i\partial_t \psi(t,x) = \Delta_x \psi(t,x) + \underbrace{f(x,\psi(t,x))}_{\text{Non linearity}} \qquad \psi : [0,T]\times \mathcal{G} \to \mathbb{C}$

for us a graph

in our case: $f(x,\psi(t,x)) = |\psi(t,x)|^{p-2}\psi(t,x) \qquad p>2$

We are interested in the so called <u>standing waves</u> solutions:

$$\psi(t,x) = e^{i\lambda t}u(t) \qquad \lambda \in \mathbb{R} \quad \text{given parameter}$$

Substituting in (tNLS) we get

(NLS) $\quad \boxed{\lambda u(x) = \Delta_x u + |u(x)|^{p-2}u(x)} \qquad u : \mathcal{G} \to \mathbb{R}$

time independent

<u>Physical motivation</u> Bose Einstein condensates: a very diluted collections of N Bosons at very low temperatures $\to$ it can be described by the <u>Nonlinear Schrödinger equation</u> instead of a system of <u>N linear equations</u> (where N is a huge number, computationally too demanding) (Gross-Pitaevskii theory)

cfr: • caso time-dependent: Erdös, Schlein, Yau, Inv. Math 2007
JAMS 2009
Ann. Math 2010

• caso stazionario: Lieb, Seiringer, Yngvason, Phys. Rev. Let. 2002, 1999
Phys. Rev. A. 2000

Def A function $u \in H^1(\mathcal{G})$ is a <u>weak solution</u> to (NLS) if

$$-\int_{\mathcal{G}} \nabla u \cdot \nabla \varphi \, dx + \int_{\mathcal{G}} |u|^{p-2} u \varphi \, dx = \lambda \int_{\mathcal{G}} u \varphi \, dx \qquad \forall \varphi \in H^1(\mathcal{G})$$

We can find weak solutions to (NLS) as minimisers of the energy E:

$$\boxed{\inf_{u\in H^1_\mu(\Omega)} \left\{ E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{1}{p}\int_\Omega |u|^p dx \right\}}$$

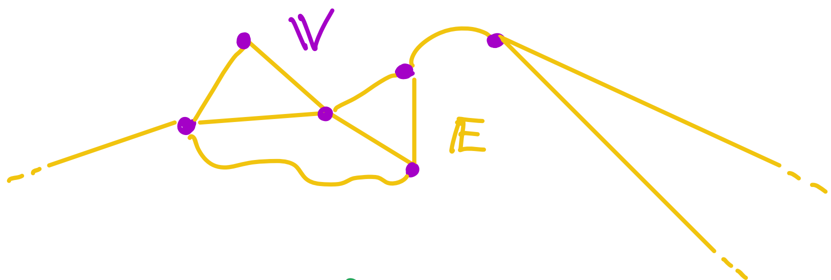
where $H^1_\mu(\Omega) := \left\{ u\in H^1(\Omega) \mid \int_\Omega |u|^2 dx = \mu \right\}$

fixed mass (otherwise $\inf = -\infty$) ($u_k(x) = k\cdot u(x)$ if $p>2$)

constrained minimum problem $\to$ Lagrange multiplier λ in the definition of weak solution

For us $\Omega = \mathcal{G}$ metric graph

Def A metric graph $\mathcal{G}$ is a graph, i.e. $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ where $\mathbb{V}$ is the set of vertices and $\mathbb{E}$ is the set of edges, $\mathbb{E} \subseteq \mathbb{V}\times\mathbb{V}$, where a coordinate $x_e : I_e := [0,\ell_e) \subseteq \bar{\mathbb{R}} \to e$ is defined on every edge $e\in\mathbb{E}$



Def $L^p(\mathcal{G}) := \left\{ u : \mathcal{G}\to\mathbb{R} \mid \|u\|^p_{L^p(\mathcal{G})} := \sum_{e\in\mathbb{E}} \int_{I_e} u|_e \, dx_e < +\infty \right\}$

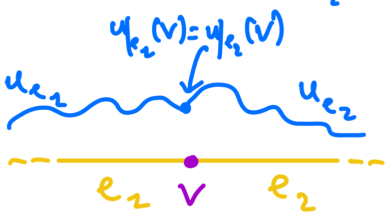
Def $H^1(\mathcal{G}) := \left\{ u : \mathcal{G}\to\mathbb{R} \mid \|u\|_{H^1(\mathcal{G})} < +\infty, \ u \text{ continuous} \right\}$

$u|_e \in H^1(I_e) \hookrightarrow C(I_e)$ + continuity in the vertices:

Sobolev embedding in 1-dimension

i.e. if $\exists\, e_1, e_2 \in \mathbb{E}$ such that $e_1 \succ v$, $e_2 \succ v$ for some $v\in\mathbb{V}$, then $u|_{e_1}(v) = u|_{e_2}(v)$

enters in



On metric graphs the notion of solution is:

Def $u\in H^1(\mathcal{G})$ solves (NLS) on $\mathcal{G}$ if: 1) $u'' + |u|^{p-2}u = \lambda u$ weakly on every edge $e\in\mathbb{E}$

2) $\sum_{e\succ v} u|_e'(v) = 0 \quad \forall v\in\mathbb{V}$

Kirchhoff condition

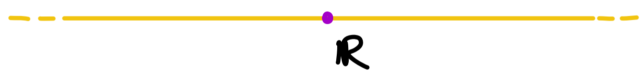
if $\mathcal{G} = [0,1]$ we also have boundary conditions (natural) (Neumann homogeneous)

__Physical motivation__ Whenever in a physical experiment a ramified structure is involved (e.g. in propagation of signals, in trapping a Boson gas) a graph appears.

II. NLS on $\mathbb{R}$: the role of the Gagliardo-Nirenberg inequalities (GL)

Ref: R. Adami, E. Serra, P. Tilli, Nonlinear dynamics on branched structures and network

The problem we want to study is:



$$\boxed{\mathcal{E}_{\mathbb{R}}(\mu) := \inf_{u\in H^1_\mu(\mathbb{R})} \left\{ E(u) := \frac{1}{2}\|u'\|^2_{L^2(\mathbb{R})} - \frac{1}{p}\|u\|^p_{L^p(\mathbb{R})} \right\}} \quad \overset{?}{>} -\infty$$

The minimizers are called ground states

Since graphs are locally 1-dimensional, we have that, $\forall\, \mathcal{G}$, the 1-dim Sobolev

(S1) $$\boxed{\|u\|_{L^\infty(\mathcal{G})} \lesssim \|u'\|_{L^2(\mathcal{G})}}$$

which implies the 1-dimensional GL inequality.

(GL1) $$\boxed{\|u\|^p_{L^p(\mathcal{G})} \lesssim \|u\|^{\frac{p}{2}+1}_{L^2(\mathcal{G})} \|u'\|^{\frac{p}{2}-1}_{L^2(\mathcal{G})}}$$

Let's look what this implies for the energy when $\mathcal{G} = \mathbb{R}$: → w.l.o.g. for graphs with half lines

$$E(u) := \frac{1}{2}\|u'\|^2_{L^2(\mathbb{R})} - \frac{1}{p}\|u\|^p_{L^p(\mathbb{R})} \geq \frac{1}{2}\|u'\|^2_{L^2(\mathbb{R})} - \frac{c}{p}\underbrace{\|u\|^{\frac{p}{2}+1}_{L^2(\mathbb{R})}}_{=\mu^{\frac{p}{4}+\frac{1}{2}}}\|u'\|^{\frac{p}{2}-1}_{L^2(\mathbb{R})}$$

- $\boxed{p>6}$ $\Rightarrow \frac{p}{2}-1>2 \Rightarrow \mathcal{E}_{\mathbb{R}}(\mu) = -\infty \Rightarrow \nexists$ minimizers $\forall \mu$!
- $\boxed{2<p<6}$ $\Rightarrow \frac{p}{2}-1<2 \Rightarrow \mathcal{E}_{\mathbb{R}}(\mu) > -\infty$ and $E(u_n) \leq C \Rightarrow \|u_n\|_{H^1(\mathbb{R})} \leq C'$

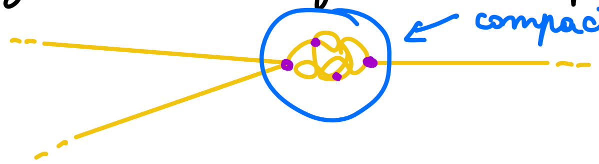
$\Rightarrow$ Given an infimizing sequence $(u_n)_n$, we have $u_n \rightharpoonup u \in H^1(\mathbb{R})$

To have u ground state we need: • $u \in H^1_\mu(\mathbb{R})$

• $u_n \to u$ $L^p(\mathbb{R})$ strongly

For $\mathbb{R}$ these holds $\Rightarrow$ for $2<p<6$ $\exists$ ground states on $\mathbb{R}$ $\forall \mu$

(For $\mathcal{G}$ graph with half lines, it depends on the topology of $\mathcal{G}$)



- $\boxed{p=6}$ $\Rightarrow \frac{p}{2}-1=2 \Rightarrow \|u\|^6_{L^6(\mathcal{G})} \lesssim \|u\|^4_{L^2(\mathbb{R})} \|u'\|^2_{L^2(\mathbb{R})}$

$$E(u) := \frac{1}{2}\|u'\|^2_{L^2(\mathbb{R})} - \frac{1}{p}\|u\|^6_{L^6(\mathbb{R})} \geq \frac{1}{2}\|u'\|^2_{L^2(\mathbb{R})} - \frac{K_6}{6}\underbrace{\|u\|^4_{L^2(\mathbb{R})}}_{=\mu^2}\|u'\|^2_{L^2(\mathbb{R})}$$

$$= \frac{1}{2}\|u'\|^2_{L^2(\mathbb{R})}\underbrace{\left(1-\frac{K_6}{3}\mu^2\right)}_{\text{depends on } \mu} > -\infty$$

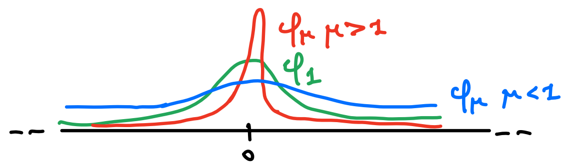
$$\Rightarrow \mathcal{E}_{\mathbb{R}}(\mu) = \begin{cases} 0 & \text{if } \mu \leq \sqrt{\frac{3}{K_6}} \\ -\infty & \text{if } \mu > \sqrt{\frac{3}{K_6}} \end{cases}$$

and ground states exist iff $\mu = \sqrt{\frac{3}{K_6}}$

Moreover, in $\mathbb{R}$, when ground states exist, they have a very specific form

$\varphi_\mu(x) = \mu^\alpha \varphi_1(\mu^\beta x)$ (Solitons)

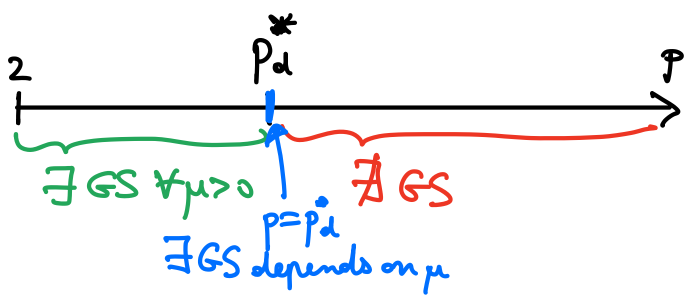
$\alpha = \frac{2}{6-p}$ $\quad \beta = \frac{p-2}{6-p}$ $\quad \varphi_1(x) = C_p \operatorname{sech}^{\frac{2}{p-2}}(c_p x)$



In general in $\mathbb{R}^d$ we have the following:

Theorem ($\exists$ ground states in $\mathbb{R}^d$) Let $p_d^* = \frac{4}{d}+2$. Then

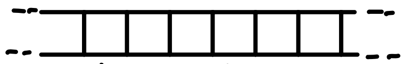
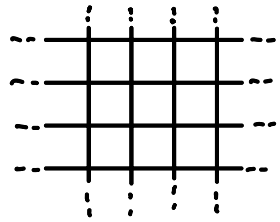
- if $2<p<p_d^*$, $\exists$ ground states
- if $p>p_d^*$, $\nexists$ ground states
- if $p=p_d^*$, there exists $\mu_d^*>0$ such that $\mathcal{E}_{\mathbb{R}^d}(\mu) = \begin{cases} 0 & \mu \leq \mu_d^* \\ -\infty & \mu > \mu_d^* \end{cases}$
  and $\exists$ ground states iff $\mu = \mu_d^*$



II. NLS on doubly periodic metric graphs

Ref: R. Adami, S. Dovetta, E. Serra, P. Tilli, Dimensional crossover with a continuum of critical exponents for NLS on doubly periodic metric graphs

Intuitive idea

- Ladder graph  $\mathbb{Z}$-periodic
- Square lattice  $\mathbb{Z}^2$-periodic

As above, (S1) and hence (GL1) still holds. We hence still have

- $\boxed{p>6}$ $\Rightarrow \frac{p}{2}-1>2 \Rightarrow \mathcal{E}_{\mathbb{R}}(\mu)=-\infty \Rightarrow \nexists$ minimizers $\forall \mu$!
- $\boxed{2<p<6}$ $\Rightarrow \frac{p}{2}-1<2 \Rightarrow \mathcal{E}_{\mathbb{R}}(\mu)>-\infty$ and $E(u_n)\leq C \Rightarrow \|u_n\|_{H^1(\mathbb{R})}\leq C'$

Theorem ($\mathbb{Z}$-periodic graphs) $\forall\, \mathcal{G}$ $\mathbb{Z}$-periodic graph, $\forall p\in(2,6)$, $\forall \mu>0$, there exist ground states. $\longrightarrow$ purely one-dimensional behaviour

For $\mathbb{Z}^2$-periodic graphs we have a more interesting structure:

Theorem ($\mathbb{Z}^2$-periodic graphs) We have two regimes:

- If $2<p<4$, then $\exists$ ground states in $H^1_\mu(\mathcal{G})$ $\forall \mu>0$
- If $4\leq p<6$, then there exists a critical mass value $\mu_p^*>0$ such that:
  - if $\mu>\mu_p^*$, then $\exists$ ground states in $H^1_\mu(\mathcal{G})$ $\rightarrow$ large masses
  - if $\mu<\mu_p^*$, then $\nexists$ ground states in $H^1_\mu(\mathcal{G})$ $\rightarrow$ small masses

The two relevant exponent are :
- $p=p_1^*=6$ $\rightarrow$ critical exponent for $d=1$
- $p=p_2^*=4$ $\rightarrow$ critical exponent for $d=2$

The presence of $p_2^*=4$ depends on the fact that the square lattice Q has a bidimensional nature. In general in $\mathbb{R}^2$ we have the 2-dim Sobolev ineq $\|u\|_{L^2(\mathbb{R}^2)}\lesssim\|\nabla u\|_{L^1(\mathbb{R}^2)}$

In Q we have

(S2) $\boxed{\|u\|_{L^2(Q)}\lesssim\|u'\|_{L^1(Q)}}$

and hence the 2-dim GL inequality

$$(GL2) \qquad \boxed{\|u\|^p_{L^p(Q)} \lesssim \|u\|^2_{L^2(Q)} \|u'\|^{p-2}_{L^2(Q)}}$$

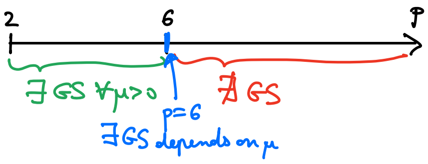
Combining (GL2) with (GL1) we get

$$\|u\|^p_{L^p(Q)} \lesssim \|u\|^{p-2}_{L^2(Q)} \|u'\|^2_{L^2(Q)}$$

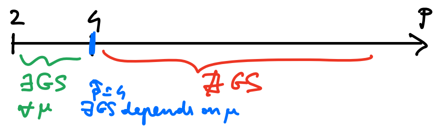
→ same exponent as first term in $E(u)$

To sum up:

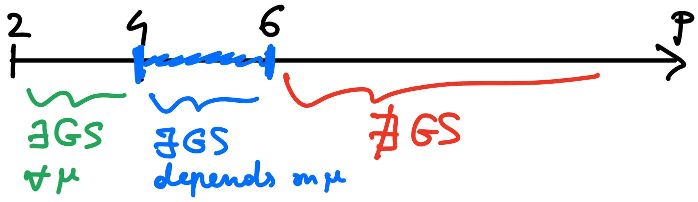
• Only (GL1):



• Only (GL2)

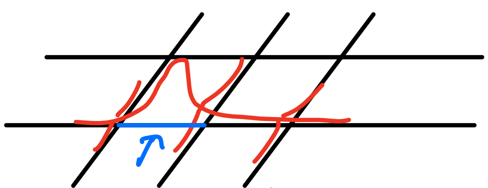


• (GL1) + (GL2)



∃ GS for large masses → the 1-dim local nature prevails

∄ GS for small masses → the 2-dim global nature prevail



By analogy with $\mathbb{R}$, we expect the ground states to be "like Solitons", which for large μ are mostly concentrated on one edge → local → basically 1-dim behaviour → for $p<6$ in $\mathbb{R}$ ∃ ground states

→ On the other hand, for small μ, the soliton will be "spread out" → the global bidimensional nature of the lattice will prevail → in $\mathbb{R}^2$ for $p>6$ there exist no ground states